Kuratowski limits of subsets of real line and their applications to pretangent spaces

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Let (X,d) be an unbounded metric space and $\tilde{r}=(r_n)_{n\in\mathbb{N}}$ be a scaling sequence of positive real numbers tending to infinity. We define the pretangent and tangent spaces $\Omega^X_{\infty,\tilde{r}}$ to (X,d) at infinity as metric spaces whose points are equivalence classes of sequences $(x_n)_{n\in\mathbb{N}}\subset X$ which tend to infinity with the speed of $\tilde{r}$. The detailed description of constructions of these spaces and their basic properties see, e. g., in [2].

Let (Y,δ) be a metric space. For any sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty sets $A_n\subseteq Y$, the *Kuratowski limit inferior* of $(A_n)_{n\in\mathbb{N}}$ is the subset $\mathop{Li}\limits_{n\to\infty} A_n$ of Y defined by the rule:

$$\left(y\in \mathop{Li}\limits_{n\to\infty} A_n\right)\Leftrightarrow(\forall\varepsilon>0\ \exists n_0\in\mathbb{N}\ \forall n\geq n_0\colon B(y,\varepsilon)\cap A_n\neq\varnothing),$$

where $B(y,\varepsilon)$ is the open ball of radius $\varepsilon>0$ centered at the point $y\in Y$,

$$B(y,\varepsilon)=\{x\in Y:\delta(x,y)<\varepsilon\}.$$

Similarly, the *Kuratowski limit superior* of $(A_n)_{n\in\mathbb{N}}$ can be defined as the subset $\mathop{Ls}\limits_{n\to\infty} A_n$ of Y for which

$$\left(y\in \mathop{Ls}\limits_{n\to\infty} A_n\right)\Leftrightarrow(\forall\varepsilon>0\ \forall n\in\mathbb{N}\ \exists n_0\geq n: B(y,\varepsilon)\cap A_{n_0}\neq\varnothing).$$

The Kuratowski limit inferior and limit superior are basic concepts of set-valued analysis in metric spaces and have numerous applications (see, for example, [1]).

We denote $tA:=\{tx:x\in A\}$ for any nonempty set $A\subseteq\mathbb{R}$ and $t\in\mathbb{R}$, and, $\nu_0:=\tilde{X}^0_{\infty,\tilde{r}}\in\Omega^X_{\infty,\tilde{r}}$ for any pretangent space $\Omega^X_{\infty,\tilde{r}}$ of an unbounded metric space (X,d). Moreover, for every scaling sequence $\tilde{r}$, we denote by $\mathbf{\Omega^X_{\infty,\tilde{r}}}$ the set of all pretangent at infinity spaces to (X,d) with respect to $\tilde{r}$. Write

$$Sp\left(\Omega^X_{\infty,\tilde{r}}\right):=\{\rho(\nu_0,\nu)\colon \nu\in\Omega^X_{\infty,\tilde{r}}\}\text{ and } Sp(X):=\{d(p,x)\colon x\in X\}.$$

Твердження 1. *Let (X,d) be an unbounded metric space, $p\in X$, $\tilde{r}=(r_n)_{n\in\mathbb{N}}$ be a scaling sequence and let $\tilde{\mathbf{R}}$ be the set of all infinite subsequences of $\tilde{r}$. Then the equalities*

$$\bigcup_{\Omega^X_{\infty,\tilde{r}}\in\mathbf{\Omega^X_{\infty,\tilde{r}}}} Sp\left(\Omega^X_{\infty,\tilde{r}}\right)=\mathop{Li}\limits_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right),$$

$$\bigcup_{\Omega^X_{\infty,\tilde{r}'}\in\mathbf{\Omega}^X_{\infty,\tilde{r}'},\,\tilde{r}'\in\tilde{\mathbf{R}}} Sp(\Omega^X_{\infty,\tilde{r}'})=\mathop{Ls}\limits_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right)$$

hold.

Наслідок 2. *Let (X,d) be an unbounded metric space, $\tilde{r}$ be a scaling sequence and let ${}^1\Omega^X_{\infty,\tilde{r}}$ be tangent and separable. Then we have*

$$\mathop{Li}\limits_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right)=\mathop{Ls}\limits_{n\to\infty}\left(\frac{1}{r_n}Sp(X)\right)=Sp\left({}^1\Omega^X_{\infty,\tilde{r}}\right).$$

Наслідок 3. *Let (X,d) be an unbounded metric space, $\tilde{r}$ be a scaling sequence. Then the sets*

$$\bigcup_{\Omega^X_{\infty,\tilde{r}}\in\mathbf{\Omega^X_{\infty,\tilde{r}}}} Sp(\Omega^X_{\infty,\tilde{r}}) \quad and \quad \bigcup_{\Omega^X_{\infty,\tilde{r}'}\in\mathbf{\Omega^X_{\infty,\tilde{r}'}},\,\tilde{r}'\in\tilde{\mathbf{R}}} Sp(\Omega^X_{\infty,\tilde{r}'})$$

are closed subsets of $[0,\infty)$.

Recall that a metric space (Y,δ) is said to be *strongly rigid* if for all $x,y,z,w\in Y$ the conditions $\delta(x,y)=\delta(w,z)$ and $x\neq y$ imply that $\{x,y\}=\{z,w\}$. Let us consider a strongly rigid metric space (Y,δ) such that:

(i_1) $\delta(x,y)<2$ for all points $x,\, y\in Y$; (i_2) $\sup\{\delta(x,y):x,y\in Y\}=2$;

(i_3) The cardinality of the open ball $B(y^*,r)=\{y\in Y:\delta(y,y^*)<r\}$ is finite for every $r\in(0,2)$ and every $y^*\in Y$.

Наслідок 4. *Let (X,d) be an unbounded metric space, $\tilde{r}$ be a scaling sequence, $\Omega^X_{\infty,\tilde{r}}$ be tangent and let (Y,δ) be a strongly rigid metric space satisfying conditions (i_1)-(i_3). If $Y_1\subseteq Y$ and $f:\Omega^X_{\infty,\tilde{r}}\to Y_1$ is an isometry, then $\Omega^X_{\infty,\tilde{r}}$ is finite.*

Приклад 5. Let (Y,δ) be a metric space with $Y=\mathbb{N}$ and the metric δ defined such that:

$$\begin{gathered}\delta(1,2)=1+\frac{1}{2};\\ \delta(1,3)=1+\frac{2}{3},\quad \delta(2,3)=1+\frac{3}{4};\\ \delta(1,4)=1+\frac{4}{5},\quad \delta(2,4)=1+\frac{5}{6},\quad \delta(3,4)=1+\frac{6}{7};\\ \delta(1,5)=1+\frac{7}{8},\quad \delta(2,5)=1+\frac{8}{9},\quad \delta(3,5)=1+\frac{9}{10},\quad \delta(4,5)=1+\frac{10}{11};\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .\end{gathered}$$

Then (Y,δ) is a countable, complete and strongly rigid metric space satisfying conditions (i_1)-(i_3). By Corollary 4 no tangent space $\Omega^X_{\infty,\tilde{r}}$ is isometric to (Y,δ).

Наслідок 6. *Let (X,d) be an unbounded metric space and let $\tilde{r}$ be a scaling sequence. Then the following statements are equivalent:*

(i) *There is a single-point pretangent space $\Omega^X_{\infty,\tilde{r}}$;*

(ii) *All $\Omega^X_{\infty,\tilde{r}}$ are single-point;*

(iii) *The equality*

$$\underset{n\to\infty}{Li}\left(\frac{1}{r_n}Sp(X)\right)=\{0\}$$

holds.

Література

[1] J.-P. Aubin, H. Frankowska. *Set-valued Analysis*, Birkhäuser, Boston, Basel, Berlin: 1990.

[2] Viktoriia Bilet, Oleksiy Dovgoshey. Finite asymptotic clusters of metric spaces. *Theory and Applications of Graphs*, 5(2) : 1–33, 2018.